

Exam I , MTH 221, Summer 2018

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Score = $\frac{57}{68}$

QUESTION 1. (a)(**8 points**) Find the solution set of the following system

$$x_2 + x_3 - x_4 + x_5 = 4, \quad x_1 - x_2 + x_3 + x_4 + 3x_5 = 6, \quad -x_1 + x_2 - x_3 - x_4 - 3x_5 = -6$$

$\boxed{3\times5}$

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & c \end{array}$$

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 1 & 4 \\ 1 & -1 & 1 & 1 & 3 & 6 \\ -1 & 1 & -1 & -1 & -3 & -6 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 1 & 4 \\ 1 & 0 & 2 & 0 & 4 & 10 \\ -1 & 0 & -2 & 0 & -4 & -10 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 1 & 4 \\ 1 & 0 & 2 & 0 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = -x_3 + x_4 - x_5 + 4$

$x_1 = -2x_3 - 4x_5 + 10$

+ leading variables : x_1, x_2
+ free variables : $x_3, x_4, x_5 \in \mathbb{R}$

Sol Set $\{(-2x_3 - 4x_5 + 10, \ -x_3 + x_4 - x_5 + 4, \ x_3, x_4, x_5) \mid x_3, x_4, x_5 \in \mathbb{R}\}$

(b)(**1 point**) Can we write the solution set of the system in (a) as span of some points? Explain

+ No, we cannot write it as span. Span is always written for homogeneous equation. This system of L.E. is clearly non homogeneous & doesn't have solution of (0,0,0,0,0)

(c) (**1 points**) Give me one point that is in the solution set of the system in (a).

$x_3 = 1$
$x_4 = 1$
$x_5 = 1$

(4, 3, 1, 1, 1)

QUESTION 2. Let A be a 5×5 matrix such that the second column of A is identical to the fifth column of A. Let B be the second column of A and consider the system of linear equations $AX = B$.

a)(**3 points**) Convince me that the system is consistent by giving me a point that is in the solution set.

$AX = B$

$[\ A\ ] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [\ B\ ]$

(0, 1, 0, 0, 0)

b) (**2 points**) Convince me that the system has infinitely many solutions by giving me three points that are in the solution set of the system.

B consistent for its linear combination of columns of A with scalars from X

$\{(0,1,0,0,0), \ (0,0,0,0,1), \ (0,-1,0,0,2)\}$

other point

(0,1,0,0,-2)

c)(**3 points**) Assume that the numbers : 1, 6, 2, -3, 9 are on the main diagonal of A. Can you find $|A|$? if yes, then find $|A|$. If no, then explain.

Yes: Since the matrix has a solution set (consistent) and has infinitely many solution. $\det(A) = 0$

QUESTION 3. (**6 points**) Let $A = \begin{bmatrix} a & b & c \\ 1 & e & 2 \\ 2 & h & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2h & 6 \\ 1 & e & 2 \\ a & (b+3) & c \end{bmatrix}$. Given $|A| = 6$. Find $|B|$(Hint: Stare well and use the definition of determinant)

Column 2

$|A| = (b)(-1)^3 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + e(-1)^4 \begin{vmatrix} a & c \\ 2 & 3 \end{vmatrix} + h(-1)^5 \begin{vmatrix} a & c \\ 1 & 2 \end{vmatrix}$

$|A| = -b(-1) + e(3a - 2c) - h(2a - c) \sim b + e(3a-2c) - h(2a-c)$

$|B| = (2h)(-1)^3 \begin{vmatrix} 1 & 2 \\ a & c \end{vmatrix} + e(-1)^4 \begin{vmatrix} 4 & 6 \\ a & c \end{vmatrix} + (b+3)(-1)^5 \begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix}$

$|B| = -2h(c - 2a) + e(4c - 6a) - 1(b+3)(2)$

$|B| = -2[-h(2a-c) + e(3a-2c) + b + 3]$

$|B| = -2[|A| + 3]$

$|B| = -2[(6) + 3]$

$\boxed{|B| = -18}$

QUESTION 4. (a) **(6 points)** Let $A = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 5 & -8 \\ -4 & -19 & 4 \end{bmatrix}$. If possible find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 4 & 1 & 0 & 0 \\ 1 & 5 & -8 & 0 & 1 & 0 \\ -4 & -19 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{1}{4} & 0 & 0 \\ 1 & 5 & -8 & 0 & 1 & 0 \\ -4 & -19 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-4R_1+R_3 \to R_3]{+8R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{1}{4} & 0 & 0 \\ 1 & 5 & 0 & 2 & 1 & 0 \\ -4 & -19 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{4R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{1}{4} & 0 & 0 \\ 1 & 5 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 7 & 4 & 1 \end{array}\right] \xrightarrow{-5R_3+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{1}{4} & 0 & 0 \\ 1 & 0 & 0 & -33 & -19 & -5 \\ 0 & 1 & 0 & 7 & 4 & 1 \end{array}\right] \xrightarrow{\text{interchange rows}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -33 & -19 & -5 \\ 0 & 1 & 0 & 7 & 4 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & 0 & 0 \end{array}\right]$$

A^{-1}

A^{-1} exists, $A^{-1} = \begin{bmatrix} -33 & -19 & -5 \\ 7 & 4 & 1 \\ 1/4 & 0 & 0 \end{bmatrix}$

(b) **(4 points)** Let $A = \begin{bmatrix} -2 & a \\ b & 2 \end{bmatrix}$. Given that $A = A^{-1}$. Find a possible INTEGER values for a and a possible INTEGER value for b.

$A = \begin{bmatrix} -2 & a \\ b & 2 \end{bmatrix}$ $\qquad A^{-1} = \frac{1}{|A|}\begin{bmatrix} 2 & -a \\ -b & -2 \end{bmatrix}$ $\qquad \boxed{\begin{array}{l} a = -1 \\ b = 3 \end{array}}$

$|A| = -4 - ab$

$-4 - ab = -1$ $\qquad -ab = 3$

$\boxed{ab = -3}$

QUESTION 5. (8 points) Write the solution set of the following homogenous system as span of INDEPENDENT points.

$2x_1 + 2x_2 + 4x_3 - 6x_4 = 0, \quad x_1 + x_2 + 2x_3 - 2x_4 = 0, \quad -x_1 - x_2 - 2x_3 + 3x_4 = 0$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & C \end{array}$$
$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & -6 & 0 \\ 1 & 1 & 2 & -2 & 0 \\ -1 & -1 & -2 & 3 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 0 \\ 1 & 1 & 2 & -2 & 0 \\ -1 & -1 & -2 & 3 & 0 \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{+3R_2+R_1 \to R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

+ leading variables : x_1, x_4
+ free variables : x_2, x_3

$x_1 = -x_2 - 2x_3$
$x_4 = 0$

$\{(-x_2 - 2x_3, x_2, x_3, 0) \mid x_2, x_3 \in \mathbb{R}\}$

* 2 independent points

* $\{C_1Q_1 + C_2Q_2\} \approx \{x_2(-1,1,0,0) + x_3(-2,0,1,0)\}$

Sol set = Span $\{(-1,1,0,0), (-2,0,1,0)\}$

* independent points

QUESTION 6. (3 points) Are the points $(1, 0, 0, \cancel{0})$, $(2, 2, 0)$, $(5, 3, 2)$ independent points? explain

+ $C_1(1,0,0) + C_2(2,2,0) + C_3(5,3,2) = (0,0,0)$

$(C_1, 0, 0) + (2C_2, 2C_2, 0) + (5C_3, 3C_3, 2C_3) = (0,0,0)$

$C_1 + 2C_2 + 5C_3 = 0$ $\qquad \boxed{C_1 = 0}$

$2C_2 + 3C_3 = 0$ $\qquad \boxed{C_2 = 0}$

$2C_3 = 0$

$\boxed{C_3 = 0}$

* Since the scalars used are all 0's for the linear combination of scalars and points to yield $(0,0,0) \rightarrow$ the points are independent

QUESTION 7. Consider the following system of equations:

$$ax_1 + x_2 - bx_3 + x_4 = 12, \quad -ax_1 - x_2 + bx_3 + 5x_4 = 0, \quad 2ax_1 + 2x_2 - 2bx_3 + cx_4 = 0$$

thinking aloud

if a any IR $\neq 0$

$$\left[\begin{array}{cccc|c} a & 1 & -b & 1 & 12 \\ -a & -1 & b & 5 & 0 \\ 2a & 2 & -2b & c & 0 \end{array}\right]$$

R_1 + R_2 (change in R_2)
-2R_1 + R_3(Change in R_3)

a 1 -b 1 12
0 0 0 6 12
0 0 0 (c -2) -24

Read:
ax_1 + x_2 -bx3 + x_4 = 12
6x_4 = 12. Hence x_4 = 2
(c-2)x_4 = -24. Hence (c-2)(2) = -24. Thus c = -10.
Consistent only if a, b any real numbers and c = -10

a)(4 points) For what values of a, b, c will the system be consistent?

$a \in \mathbb{R}$ $b \in \mathbb{R}$ $c \neq 0$ *by staring

b)(2 points) For what values of a, b, c will the system have unique solution?

NONE (free variable always exist)

c)(2 points) For what values of a, b, c will the system be inconsistent?

$a \in \mathbb{R}$ $b \in \mathbb{R}$ $\cancel{c=0}$

and c NOT EQUAL -10

QUESTION 8. (6 points) Let
$$A = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline \end{array}$$
$$A = \begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 7 & 2 & 4 \\ -4 & 8 & 0 & 2 \\ -2 & 4 & -2 & 0 \end{bmatrix}$$
. Use crammer rule to find the value of x_4 when solving the system of linear equations $AX = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

* Find det(A)=?

$$\begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 7 & 2 & 4 \\ -4 & 8 & 0 & 2 \\ -2 & 4 & -2 & 0 \end{bmatrix} \quad \begin{array}{c} R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3 \\ R_1+R_4\to R_4 \end{array} \quad \begin{bmatrix} 2 & -4 & 2 & 2 \\ 0 & 3 & 4 & 6 \\ 0 & 0 & 4 & 10 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

upper triangle

$\det(A) = (2)(3)(4)(2) = 48 \neq 0$ (unique sol)

$|\,| = |A|$

$$x_4 = \frac{\begin{vmatrix} 2 & -4 & 2 & 1 \\ -2 & 7 & 2 & 2 \\ -4 & 8 & 0 & 1 \\ -2 & 4 & -2 & 0 \end{vmatrix}}{\det(A)} \quad \begin{array}{c} R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3 \\ R_1+R_4\to R_4 \end{array} \quad \frac{\begin{vmatrix} 2 & -4 & 2 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}}{\det(A)} = \frac{(2)(3)(4)(1)}{48} = \frac{1}{2}$$

$\boxed{x_4 = \frac{1}{2}}$

QUESTION 9. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ $\begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

a) **(2 points)** Use the concept of linear combination of columns and find $Col_2(A^2)$ (second column of A^2)

$$2\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 3 \end{bmatrix}$$

b) **(2 points)** Use the concept of linear combination of rows and find $Row_3(A^2)$

$$1\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} + 1\begin{bmatrix} -3 & 2 & 2 \end{bmatrix} + -1\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 4 \end{bmatrix}$$

2

Faculty information

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